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A Well Behaved Exact Solution for Spherically Symmetric Perfect Fluid Ball

B.C. Tewari, Kali Charan and Prakash Chandra*
Department of Mathematics,
Kumaun University, SSJ Campus, Almora, India.
E-mail: drbctewari@yahoo.co.in, kcyadav2008@gmail.com
*Department of Mathematics,
Kumaun University, MBGPG College, Haldwani, India.
E-mail: pcfsouk@gmail.com

Dedicated to Prof. Hari M. Srivastava on his 75^{th} ***birth anniversary***

Abstract: We here present a spherically symmetric exact solution of the general relativistic field equations by using Tewari [1] solution as a seed solution. The solution is having positive finite central pressure and positive finite central density. The ratio of pressure and density is less than one and casualty condition is obeyed at the centre. Further, the outmarch of pressure, density and pressure-density ratio, and the ratio of sound speed to light is monotonically decreasing. The central red shift and surface red shift are positive and monotonically decreasing. Further by assuming the suitable surface density, we have constructed a compact star model with all degree of suitability.

Keywords: Exact solution, Einsteins field equations, Perfect fluid ball, Compact star.

1. Introduction:

Due to condensation and thereafter contraction of a massive gas cloud (mass less than the solar mass) a quasi static equilibrium state is reached when a resulting thermal radiation pressure together with normal hydrodynamic pressure balances the gravitational binding energy which ends up into a compact stellar object. Einstein's field equation was obtained by Schwarzschild for the interior of this static compact stellar object. The well behaved solution of Einstein's field equation can give us an idea about the interiors of massive fluid ball. The first ever two exact solution of Einstein field equation for a compact object in static equilibrium was obtained by Schwarzschild [2]. The first solution corresponds to the geometry of the space-time exterior to a static prefect fluid ball, while the other solution describes the interior geometry of a fluid sphere of constant energy-density. Tolman [3] has obtained five different types of exact solutions for static cases. The III solution corresponds to the constant density solution obtained earlier by Schwarzschild [2]. The V and VI solutions correspond to infinite density and infinite pressure at the centre, hence not considered physically viable. Thus only the IV and VII solutions of Tolman are of physical relevance. Despite the non linear character of Einsteins field equations, various exact solutions for static and spherically symmetric metric are available in the related literature.

The search for the exact solutions is of continuous interest to researcher. Buchdahl [4] proposed a famous bound on the mass radius ratio of relativistic fluid spheres which is an important contribution in order to study the stability of the fluid spheres. Delgaty-Lake [5] studied all the then existing solutions and established that Adler [6], Heintzmann [7], Finch and Skea [8], etc. do not satisfy all the well behaved conditions and also pointed out that only nine solutions are well behaved; out of which seven in curvature coordinates (Tolman [3], Patvardhav and Vaidya [9], Mehra [10], Kuchowicz [11], Matese and Whitman [12], Durgapals two solutions [13] and only two solutions (Nariai [14], Goldman [15]) in isotropic coordinates. Ivanov [16], Neeraj Pant [17], Maurya and Gupta [18], Pant et al. ([19],[20]) studied the existing well behaved solutions of Einstein field equations in isotropic coordinates. Recently we have found some exact solutions of Einsteins field equations given by Tewari [21], Tewari and Charan ([22]-[24]).

In this paper we present another new solution in spherically symmetric coordinates which is well behaved. Keeping in view of generality of solution due to Tewari [1] we present a special solution of the same and its detailed study, in order to construct a realistic model of compact star. In our present study the paper consists of seven sections. In section 2 Einstein's field equations in isotropic coordinates are given. Section 3 consists of boundary conditions for well behaved solutions. New class of solution of Einstein's field equations in isotropic coordinates is given in section 4. Section 5 stipulates the properties of this new class of solution of Einstein's field equations. In section 6 the matching conditions of interior metric of the perfect fluid with the exterior metric are given. Finally, some concluding remarks have been made in section 7.

2. Einstein's Field Equation in Isotropic coordinates

The Einstein's field equations of general relativity are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{1}$$

where $T_{\mu\nu}$, the energy momentum tensor for a perfect fluid ball is defined as

$$T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu - pg_{\mu\nu} \tag{2}$$

where ρ and p are the proper density and isotropic pressure of the fluid, u_μ time-like four-velocity vector and g^μ_ν metric tensor of space-time.

The interior space-time metric for spherically symmetric fluid distribution is given by

$$ds^2 = -B^2\{dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\} + A^2dt^2 \tag{3}$$

where A and B are functions of r only.

In view of the metric (3) and energy momentum tensor (2), the field equation (1) gives

$$\frac{8\pi G}{c^4}p = \frac{1}{B^2}\Big(\frac{B'^2}{B^2} + \frac{2B'}{rB} + \frac{2A'B'}{AB} + \frac{2A'}{rA}\Big) \tag{4}$$

$$\frac{8\pi G}{c^4}p = \frac{1}{B^2}\Big(\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{B'}{rB} + \frac{A''}{A} + \frac{2A'}{rA}\Big) \tag{5}$$

$$\frac{8\pi G}{c^2}\rho = -\frac{1}{B^2}\Big(\frac{2B''}{B} - \frac{B'^2}{B^2} + \frac{4B'}{rB}\Big) \tag{6}$$

The gravitational redshift of massive spherically symmetric ball is

$$1 + Z = g_{00}^{\frac{1}{2}} \tag{7}$$

which gives central (Z_0) and surface (Z_Σ) gravitational redshifts

$$Z_0 = \frac{c}{A} - 1 \tag{8}$$

and

$$Z_\Sigma = \Big(1 + \frac{rB'}{B}\Big)^{-1} - 1 \tag{9}$$

3. Boundary conditions for well behaved Solution

For well behaved nature of the solution in isotropic coordinates, the following conditions should be satisfied (Bonnor-Vickers [25]):

(i) The solution should be free from singularities i.e. central pressure, central density, should be positive and finite. For this A and B must be positive or $\rho_{r=0} > 0$ and $p_{r=0} > 0$.

(ii) The solution should have positive and monotonically decreasing expressions for pressure and density with the increase of r. The solution should have positive value of ratio of pressure-density and less than 1 (weak energy condition) and less than (strong energy condition) throughout within the star, monotonically decreasing as well. (Pant and Negi [26]).

(iii) The casualty condition i.e. velocity of sound should be less than that of light throughout the model. The velocity of sound should be decreasing towards the surface and increasing with the increase of density. In this context it is worth mentioning that the equation of state at ultra-high distribution has the property that the sound speed is decreasing outwards. (Canuto and Lodenquai [27]).

(iv) $\frac{p}{\rho} < \frac{dp}{d\rho}$, everywhere within the ball. For realistic matter $\gamma > 1$. (Pant and Maurya [28])

(v) The red shift z should be positive, finite and monotonically decreasing in nature with the increase of r.

Under these conditions, we have to assume the one of the gravitational potential component in such a way that the field equation (1) can be integrated and solution should be well behaved.

4. New class of well behaved Solution

In view of eq. (4) and (5), the condition of pressure isotropy reduces to a differential equation in A and B

$$\frac{A''}{A} + \frac{B''}{B} = \Big(\frac{2B''}{B} + \frac{1}{r}\Big)\Big(\frac{A'}{A} + \frac{B'}{B}\Big) \tag{10}$$

A new parametric class of solutions of (10) is obtained by Tewari [1] as follows

$$A = D_1(1 + c_1 r^2)^{\frac{2-n}{l+1}+1} + D_2(1 + C_1 r^2)^{\frac{n}{l+1}} \tag{11}$$

$$B = C_2(1 + C_1 r^2)^{\frac{1}{l+1}} \tag{12}$$

where n, l, C_1, C_2, D_1 and D_2 are constants and

$$n = \frac{1}{2}\Big\{(l + 3) \pm (l^2 + 10l + 17)^{\frac{1}{2}}\Big\} \tag{13}$$

where n is real if $l \geq -5 + 2\sqrt{2}$ or $l \leq -5 - 2\sqrt{2}$.

In view of eq.(4)-(6) and eq.(11) and (12), we get expressions for pressure and density given as

$$\frac{8\pi G}{c^4}p = = \frac{4C_1}{(l+1)^2 C_2^2 (1 + C_1 r^2)^{2l+1+2}}[(l+1)(n+1) + \Big\{(l+1)(n+1) + 2n + 1\Big\}C_1 r^2$$

$$+ \frac{(l+1)(\frac{2-2n}{l+1} + 1)D_1(1 + C_1 r^2)^{\frac{2-2n}{l+1}+1}\Big\{(l+1) + (l+3)C_1 r^2\Big\}}{D_2 + D_1(1 + C_1 r^2)^{\frac{2-2n}{l+1}+1}}] \tag{14}$$

$$\frac{8\pi G}{c^2}\rho = \frac{4C_1}{(l+1)^2C_2^2}\frac{\Big\{-3(l+1)-(l+2)C_1r^2\Big\}}{(1+C_1r^2)^{\frac{2}{l+1}+2}} \tag{15}$$

In order to make a new realistic model we assume $n = -11/9$ and using (11)-(15), the metric coefficients and explicit expressions for the energy density,isotropic pressure of star are given by

$$A = D_1(1+c_1r^2)^{\frac{90}{119}} + D_2(1+C_1r^2)^{\frac{11}{119}} \tag{16}$$

$$B = C_2(1+C_1r^2)^{\frac{-9}{119}} \tag{17}$$

$$\frac{8\pi G}{c^4}p = \frac{4C_1}{(1147041C_2^2(1+C_1r^2)^{\frac{110}{l19}}}[238+121C_1r^2+\frac{79D_1(1+C_1r^2)^{\frac{79}{119}}\{119+l01C_1r^2\}}{D_2+D_1(1+C_1r^2)^{\frac{79}{l19}}}] \tag{18}$$

$$\frac{8\pi G}{c^2}\rho = \frac{4C_1(357+110C_1r^2)}{1147041C_2^2(1+C_1r^2)^{\frac{110}{l19}}} \tag{19}$$

5. Properties of new solution

The solution should be free from singularities i.e. central pressure, central density, should be positive and finite. For this A and B must be positive or $D_1 + D_2 \geq 0$ and $C_2 \geq 0$.

The central pressure and density of star are given by

$$\frac{8\pi G}{c^4}p_0 = \frac{16C_1}{9639C_2^2}\left\{\frac{81D_1+2D_2}{D_1+D_2}\right\} \tag{20}$$

$$\frac{8\pi G}{c^2}\rho_0 = \frac{4C_1}{357C_2^2} \tag{21}$$

The central value of pressure and density is positive definite if $81D_1+2D_2 > 0$ and $C_1 > 0$. For the values of D_1 and D_2 such that $\frac{4}{27}\left\{\frac{81D_1+2D_2}{D_1+D_2}\right\} \leq 1$, the central value $\frac{p_0}{C^2\rho_0} \leq 1$.

In view of equation (18) the rate of fall of pressure with radial distance from the center p' is given by

$$\frac{8\pi G}{c^4}p' = \frac{8rC_1^2(-1309+121C_1r^2)}{15166431C_2^2(1+C_1r^2)^{\frac{229}{119}}} + \frac{632rD_1C_1^2}{136497879C_2^2}$$

$$\left\{\frac{12109(1+C_1r^2)(D_2+D_1(1+C_1r^2)^{\frac{79}{119}})-(119+101C_1r^2)\Big\{(79D_1(1+C_1r^2)^{\frac{79}{119}}+31(D_1+D_2)\Big\}}{(1+C_1r^2)^{\frac{150}{119}}\Big\{D_2+D_1(1+C_1r^2)^{\frac{79}{119}}\Big\}^2}\right\} \tag{22}$$

$$p_0'' = -\frac{8C_1^2}{1147041C_2^2}\left\{\frac{810D_1^2 - 5728D_1D_2 + 99D_2^2}{(D_1+D_2)^2}\right\} \tag{23}$$

gives negative value of $(p'')_0$ for all values of D_1 and D_2 which satisfy boundary conditions. Hence pressure is maximum at the centre and monotonically decreasing.

In view of equation (19) the rate of fall of density with radial distance from the center ρ' is given by

$$\frac{8\pi G}{c^2}\rho' = \frac{8rC_1^2(-238+9C_1r^2)}{127449C_2^2(1+C_1r^2)^{\frac{229}{119}}} \tag{24}$$

$$\rho_0'' = -\frac{880C_1^2}{127449C_2^2} \tag{25}$$

which is always negative for all values of C_1 and C_2. Thus density is maximum at center and is monotonically decreasing.

Square of adiabatic sound speed at the center is given by

$$\frac{1}{c^2}(\frac{dp}{d\rho})_0 = \frac{0.0462(D_1^2 + 1.9956D_1D_2 + D_2^2)}{(D_1+D_2)^2} \tag{26}$$

The casuality condition is obeyed at the center for all values of constant satisfying the boundary conditions.

Further it is mentioned here that the boundary of the super dense star is established only when $D_1 + D_2 > 0$ and $C_2 > 0$, $C_1 > 0$.

6. Matching Conditions of Boundary

The solutions so obtained are to be matched over the boundary with Schwarzschild exterior solution

$$ds_+^2 = -\Big(1 - \frac{2GM}{c^2R}\Big)^{-1}dR^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2) + \Big(1 - \frac{2GM}{c^2R}\Big)c^2dt^2 \tag{27}$$

where M is the mass of the ball determined by external observer and R is the radial coordinate of the exterior region.

The usual boundary conditions are that the first and second fundamental forms are continuous over the boundary $r = r_\Sigma$ or equivalently $R = R_\Sigma$. Therefore

$$\Big[D_2(1+C_1r_\Sigma^2)^{\frac{90}{119}} + D_1(1+C_1r_\Sigma^2)^{\frac{11}{119}}\Big]^2 = c^2(1-2S_p) \tag{28}$$

$$R_\Sigma = r_\Sigma C_2(1+C_1r_\Sigma^2)^{\frac{-9}{119}} \tag{29}$$

$$\left(\frac{B'}{B}+\frac{1}{r}\right)r_\Sigma=(1-2S_p)^{\frac{1}{2}} \tag{30}$$

$$\frac{A'}{A}r_\Sigma=S_p(1-2S_p)^{\frac{-1}{2}} \tag{31}$$

where $S_p=\frac{GM}{c^2R_\Sigma}$.
In view of the above boundary conditions we get the values of the arbitrary constants in terms of Schwarzschild parameter S_p .

$$C_1=\frac{119[1-(1-2S_p)^{\frac{1}{2}}]}{18} \tag{32}$$

$$C_2=R_\Sigma\Big[\frac{1}{r_\Sigma}+\frac{119[1-(1-2S_p)^{\frac{1}{2}}]}{18}\Big]r_\Sigma \tag{33}$$

$$D_1=\frac{11c[1-(1-2S_p)^{\frac{1}{2}}](1-2S_p)^{\frac{1}{2}}(1+C_1r_\Sigma^2)^{\frac{-90}{119}}-9S_p(1-2S_p)(1+C_1r_\Sigma^2)^{\frac{30}{119}}}{[1-(1-2S_p)^{\frac{1}{2}}][11+90(1+C_1r_\Sigma^2)^{\frac{1}{119}}]} \tag{34}$$

$$D_2=\frac{90c[1-(1-2S_p)^{\frac{1}{2}}](1-2S_p)^{\frac{1}{2}}(1+C_1r_\Sigma^2)^{\frac{48}{119}}+9S_p(1-2S_p)(1+C_1r_\Sigma^2)^{\frac{109}{119}}}{[1-(1-2S_p)^{\frac{1}{2}}][11+90(1+C_1r_\Sigma^2)^{\frac{59}{119}}]} \tag{35}$$

The central redshift is

$$Z_0=(1-2S_p)^{\frac{-1}{2}}-1 \tag{36}$$

The surface redshift is

$$Z_\Sigma=\frac{18R_\Sigma^2[1-(1-2S_p)^{\frac{1}{2}}]}{18+101R_\Sigma^2[1-(1-2S_p)^{\frac{1}{2}}]} \tag{37}$$

7. Conclusion

We have given a new class of solution for spherically symmetric perfect fluid ball. We have obtained a variety of classes of exact solutions by giving different values to the parameter n in general solution. A new model corresponding to $n=-11/9$ has been studied in detail. It has been observed that the physical parameters pressure, density, and redshift are positive at the centre and within the limit of realistic state equation and monotonically decreasing and the causality condition is obeyed throughout the fluid ball. Thus, the solution is well behaved for all values of Schwarzschild parameter S_p within the perfect fluid ball. Our solution is useful to construct the models of compact star like Strange star family, Neutron star and many more.

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